1. The History of Numbers

In the beginning, we counted sheep. This required the natural numbers:

$$\mathbb{N} = \{1, 2, 3, ...\}.$$

This set is ordered, and is closed under addition and multiplication. This means that if we add to natural numbers, we get another natural number, and if we multiply two natural numbers, we get another natural number. What we cannot do is subtract.

As civilization advanced, we needed book keeping. Abraham owes Moses ten sheep, but Job owes Abraham seven sheep. How many sheep, in total, does Abraham have? The only reasonable answer is $7 - 10$. Thus, we invented negative numbers, to allow for subtraction:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

This set is ordered and closed under additional, multiplication, and subtraction. Unfortunately, the integers are not closed under division.

The farmer has 5 bales of hay to feed 7 sheep. How much hay should each sheep get? Clearly, he needs to divide 5 by 7. But this isn't an integer. Thus, rational numbers were invented. The rational numbers are

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

This set is closed under addition, subtraction, multiplication, and division.

It seems like $\mathbb{Q}$ is a very good set in which to do mathematics. We can perform all of the arithmetic operations that are necessary (as long as we avoid exponents). However, $\mathbb{Q}$ falls short when it comes to geometry.

The shepherd leaves home and walks one mile due east to the pasture, and later takes his sheep one mile due north to the farmer. Now he wants to go home, and realizes that he needs to walk $\sqrt{2}$ miles. But how far is this? It turns out, the distance is not rational.

Proposition 1. *$\sqrt{2}$ is not rational.*

Proof. Suppose that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ for some integers p and q. We may assume that p and q have no common factors, for if they did, we could cancel them. Then $q\sqrt{2} = p$, so $2q^2 = p^2$, so p^2 is even. This implies that p is even.

Since p is even, $p = 2k$ for some integer k. But then $2q^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$, so q^2 is even, which implies that q is even.

But we assumed that p and q had no common factors, so they cannot both be even. This contradiction shows that our assumption that $\sqrt{2} = \frac{p}{q}$ is impossible; thus, $\sqrt{2}$ is not rational. □

We see that not all distances are rational. Indeed, we know that the rational numbers are ordered, but if we place all the rational numbers in there correct place on a number line, we will not cover the entire line; there will be "gaps". We need a larger set of numbers which include all possible distances.

All distances correspond to all possible points on a line, and their negatives. There is a one-to-one correspondence between points on a line, and *decimal expansions*.

A *decimal digit* is one of these: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

A *decimal expansion* consists of an integer, followed by a decimal point, followed by an infinite sequence of digits.

We define the *real numbers* to be the set of decimal expansions:

$$\mathbb{R} = \{ \text{ decimal expansions } \}.$$

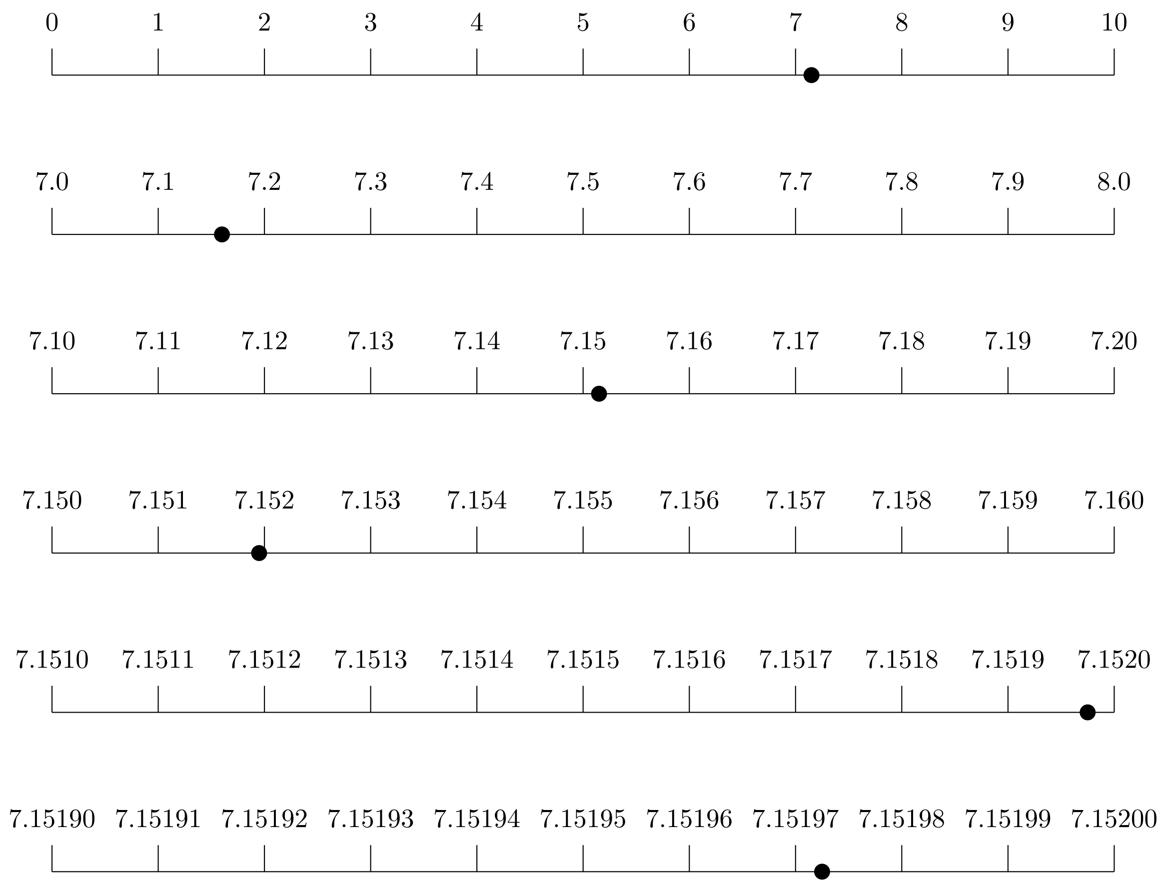
2. The Number Line

We illustrate how each point on a line corresponds to a unique decimal expansion with the following diagram.

First, we identify 0 on the line, and mark off all of the integers. Now we locate the point on the line, and see that it is between 7 and 8. Thus, the integer part of the decimal expansion is 7.

Next, we expand the line from 7 to 8 to get a closer look, and divide it into ten parts of equal size. We see that the point lies between 7.1 and 7.2, so the first digit of the decimal expansion is 1.

We again expand the line and divide the segment from 7.1 to 7.2 into ten equal parts, observe that the point is between 7.15 and 7.16, and thus obtain 5 as the next digit. Continue in this way to see that the next digits is 1, then 9, then 7, and so forth.



Continue in this way forever to obtain an infinite sequence of digits. Now we have the decimal expansion of the number which corresponds to this point.